Computational Prefractals and Their Attractors in $\mathbb{R}^2$

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Abstract

We have developed a GUI that allows users to describe geometric IFS in $\mathbb{R}^2$ and plot orbits over their attractors. Included are novel features such as sequence labeling and dimension approximation.

Iterated Function Systems and Attractors

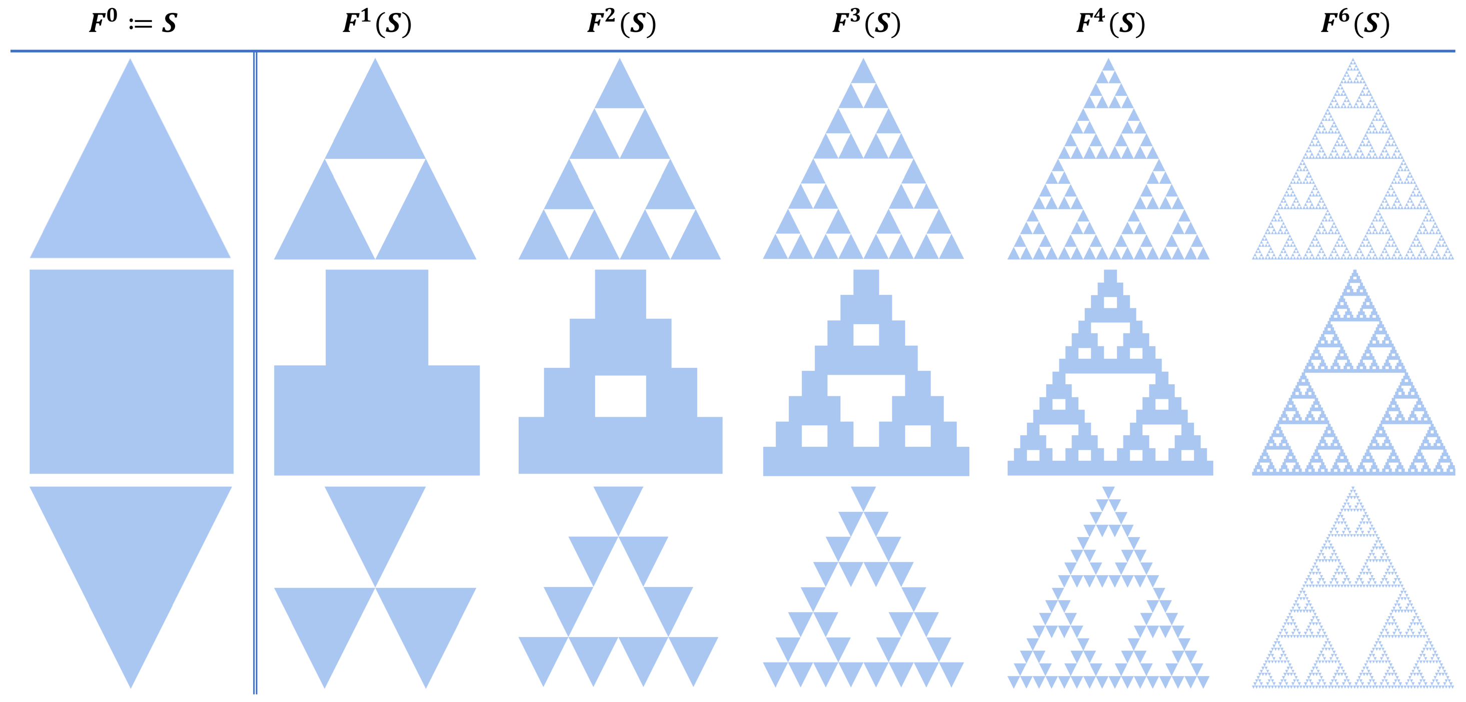
An **iterated function system** (IFS) is a collection $\mathcal{F} = \{X; f_1, \dots, f_N\}$

- X is a topological space
- $f_n: X \to X$ are continuous functions on X for $1 \le n \le N$
- H_X is the collection of nonempty compact subsets of X
- We take $\mathbb{R}^2$ with topology induced by the Hausdorff metric $d_{\mathcal{H}}$.

For $A, B \subseteq \mathbb{R}^2$, $\quad d_{\mathcal{H}}(A,B) = \inf\{\epsilon \ge 0; A \subseteq B_\epsilon \text{ and } B \subseteq A_\epsilon\}$,
where A_ϵ is the set of all points of distance ϵ from any point in A.
The **Hutchinson operator** is a continuous map $F: H_X \to H_X$

$$F(S) = \bigcup_{f \in \mathcal{F}} f(S) \qquad \text{where} \qquad F^k = \overbrace{F \circ \cdots \circ F}^{k\ \text{times}}.$$

The software allows users to select a region $S \in H_{\mathbb{R}^2}$, specify an operator F, and render $F^k(S)$ as a geometric object. We describe transformations with concise notation as strings, where $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{R}$ represent scaling, translation, and rotation. The IFS below could be described by the user as $\mathcal{S}_{.5} \circ \mathcal{T}[\mathfrak{v}]$, where $\mathfrak{v}$ is the set of vertices for the equilateral triangle.



$$\mathcal{F} = \left\{\mathbb{R}^2; \begin{bmatrix} 0.5 & 0. & 0. \\ 0. & 0.5 & 0.288675 \\ 0. & 0. & 1. \end{bmatrix}, \begin{bmatrix} 0.5 & 0. & -0.25 \\ 0. & 0.5 & -0.144337 \\ 0. & 0. & 1. \end{bmatrix}, \begin{bmatrix} 0.5 & 0. & 0.25 \\ 0. & 0.5 & -0.144337 \\ 0. & 0. & 1. \end{bmatrix}\right\}$$

In the image above we chose several $S \subset \mathbb{R}^2$ and applied the same operator F from the defined IFS $\mathcal{F}$. In all cases we converge to the Sierpiński Triangle. This is the only set $A \in H_{\mathbb{R}^2}$ such that

$$F(A) = A \qquad \text{and} \qquad \lim_{k \to \infty} F^k(S) = A \ (\text{w.r.t. } d_{\mathcal{H}}).$$

For this reason, we say that A is the **attractor** of $\mathcal{F}$.

Hutchinson's theorem states any contractive IFS on a non-empty complete metric space X has a unique attractor A, starting from any $S \subset B \subseteq X$. Where B is known as the basin of attraction. Above, $B = \mathbb{R}^2$ itself and each $f_i \in \mathcal{F}$ was contractive by a factor of ½.

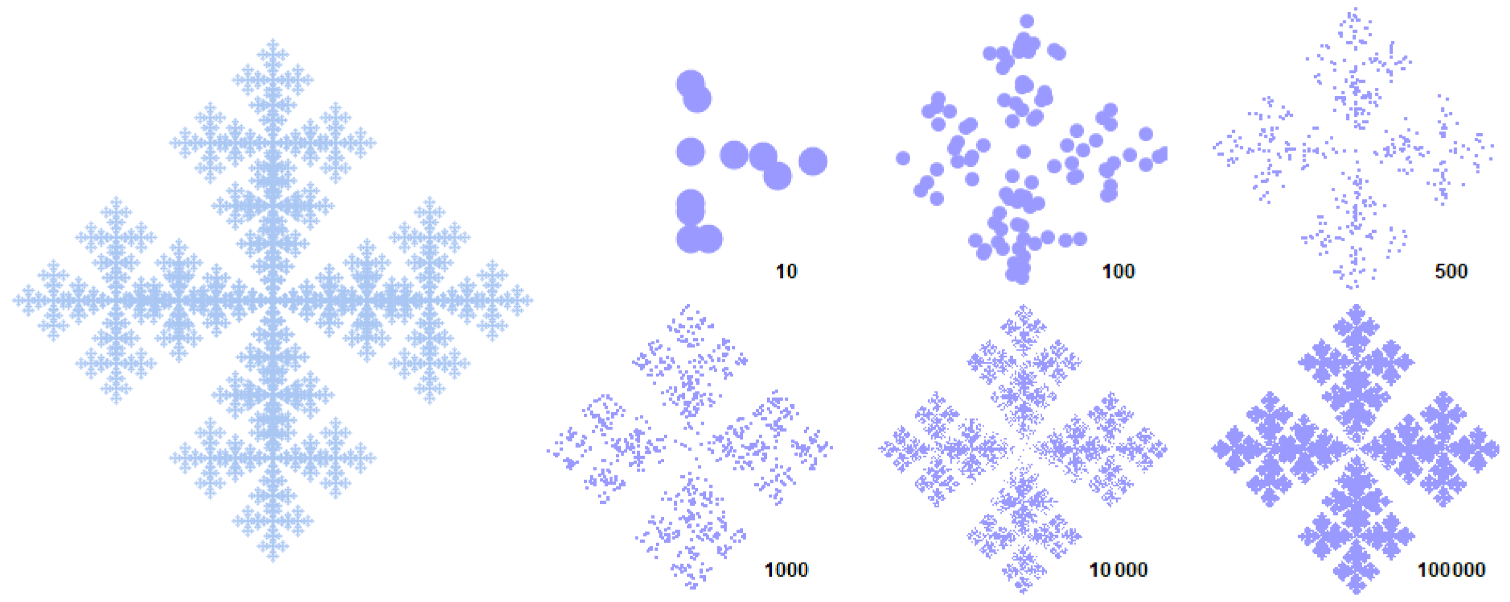
References

- Barnsley, M. F. (2012). Fractals everywhere. Mineola, NY: Dover Publications.
- Barnsley, M., & Vince, A. (2013). Developments in fractal geometry. Bulletin of Mathematical Sciences, 3(2), 299–348. doi: 10.1007/s13373-013-0041-3
- Edgar, G. A. (2011). Measure, topology, and fractal geometry. New York: Springer.

Orbits and the Chaos Game

For an IFS with n functions, computing $F^k(S)$ as a region requires n^k transformations of S. This allows us to plot prefractal sets but is resource intensive. We may plot our attractor more efficiently by plotting orbits over it using **Barnsley's chaos game algorithm.**

- Start with any seed $x_0 \in B \subseteq X$, where B is the attraction basin
- Compute $x_n := f_i(x_{n-1})$, each with a randomly selected $f_i \in \mathcal{F}$
- Eventually, the orbit will yield $A = \lim_{K \to \infty} \{x_n : n \ge K\}$ (w.r.t. $d_{\mathcal{H}}$)



The GUI uses the Wolfram language's FoldList to quickly compute the orbit $(x_k)_{k=0}^N$ for large N. The program can recompute the orbit smoothly at 10,000 points on an Intel Core i3. This allows the user to dynamically adjust parameters such as scale factor and function probabilities, changing the attractor in real time.

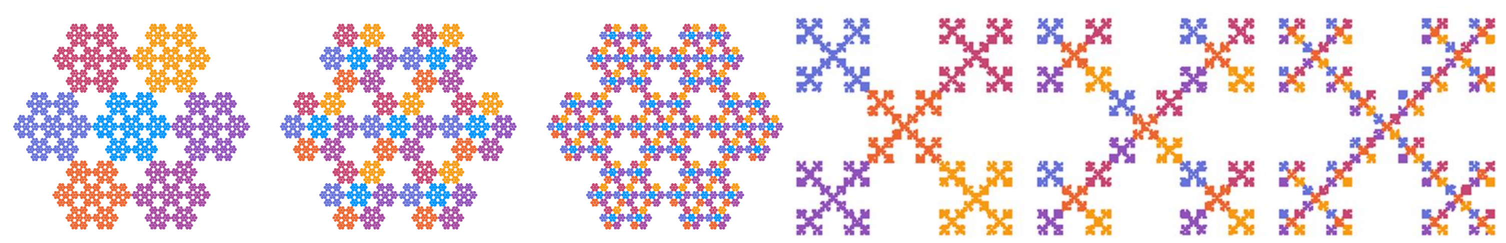
Addresses and the Sequence Space

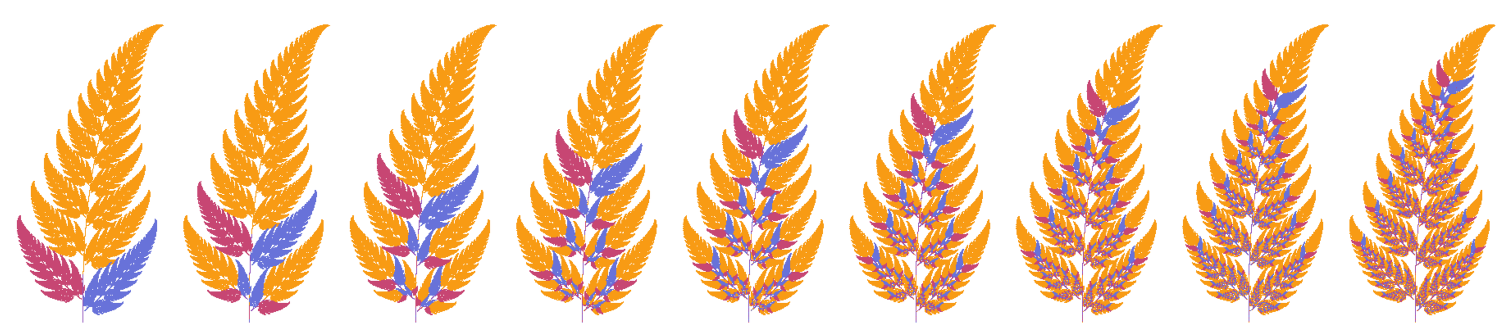
Symbolic dynamics. Consider a hypothetical IFS $\mathcal{F} = \{X; f_0, f_1\}$. We may use the fixed-point property of the attractor A to write,

- $A = f_0(A) \cup f_1(A)$
  $= f_0(f_0(A)) \cup f_0(f_1(A)) \cup f_1(f_0(A)) \cup f_1(f_1(A)) = \cdots$
- In this manner, subsets of the attractor A are associated with a finite sequence of functions $(f_{\omega_1}, f_{\omega_2}, \dots, f_{\omega_N})$.
- For a general IFS $\{X; f_1, \dots, f_N\}$, points $x \in A$ then correspond to infinite sequences of N symbols $(\omega_1, \omega_2, \dots, \omega_i, \dots)$ in the **sequence space** Ω_N, where $1 \le \omega_i \le N$ for all i.
- There is an **itinerary map** $\psi: A \to \Omega$ by $x \mapsto (\omega_1, \omega_2, \dots)$.

$$\begin{array}{ccc} A & \xrightarrow{F} & A \\ \downarrow \psi & & \downarrow \psi \\ \Omega & \xrightarrow{\sigma} & \Omega \end{array}$$

For any point x_k with $k > n$ in the chaos game orbit of A, we have developed a method to plot its n^{th} term in the sequence space using the fact that the diagram pictured to the left commutes. We use the shift map $\sigma^n((\omega_1, \omega_2, \dots)) = (\omega_{n+1}, \omega_{n+2}, \dots)$ on the history of the points orbit under F.





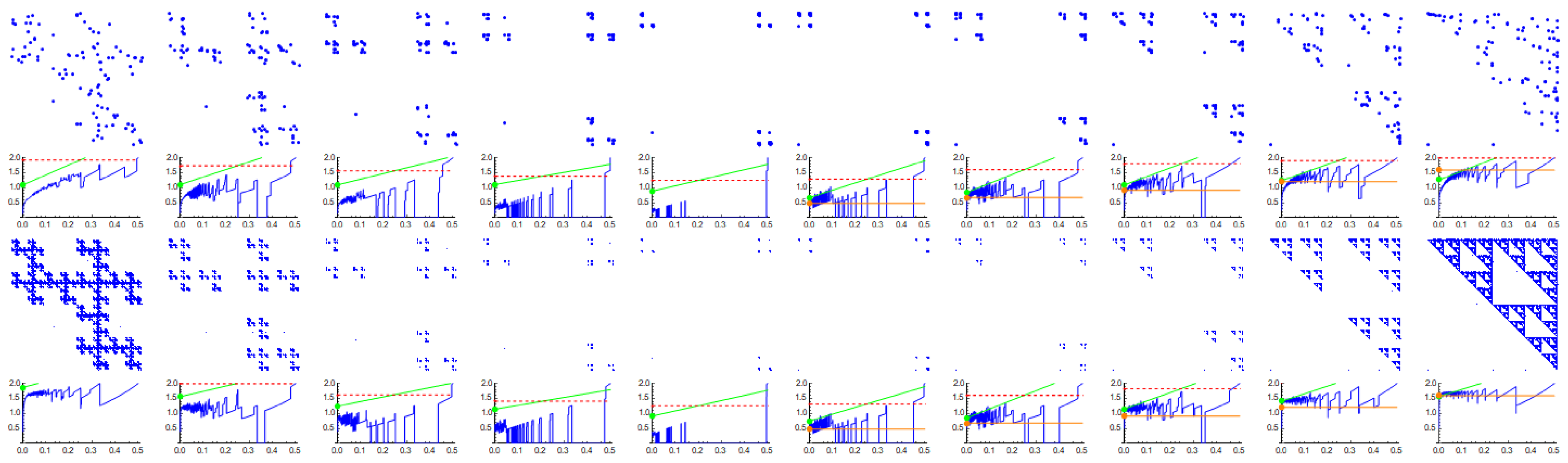
Computing Fractal Dimension of Point Clouds

In some cases, the Hausdorff dimension of an attractor, $\dim_{\mathcal{H}} A$, can be computed in exact form, but often for general IFS and in nature this is not the case. For this reason there have been several proposed fractal dimensions, the first known to provide upper bound for $\dim_{\mathcal{H}} A$. They form a non-increasing sequence indexed by α, beginning with the **box counting and information dimension**, they are listed below.

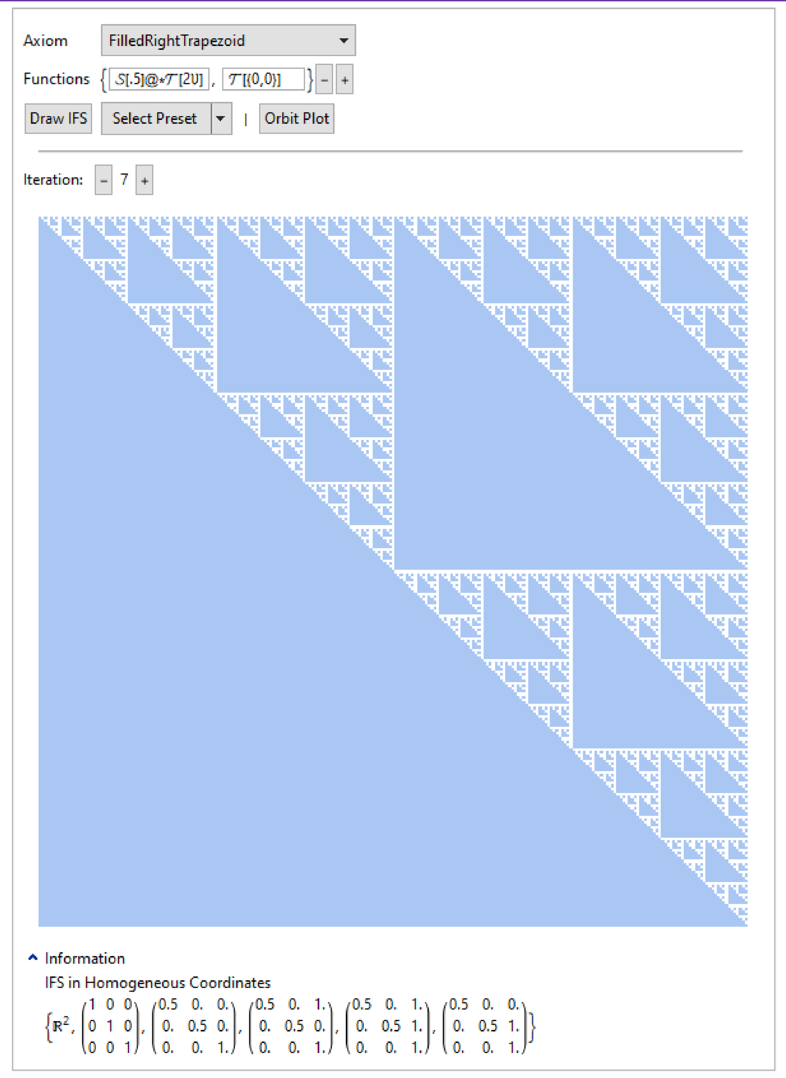
$$\dim_{Box} A := \lim_{\epsilon \to 0} \frac{\ln(N(\epsilon))}{\ln(1/\epsilon)} \ge \dim_{Info} A := \lim_{\epsilon \to 0} \frac{-\langle \ln p_\epsilon \rangle}{\ln(1/\epsilon)} \ge \cdots \ge \dim_\alpha A := \lim_{\epsilon \to 0} \frac{\mathrm{H}_\alpha(A)}{\ln(\epsilon)}$$

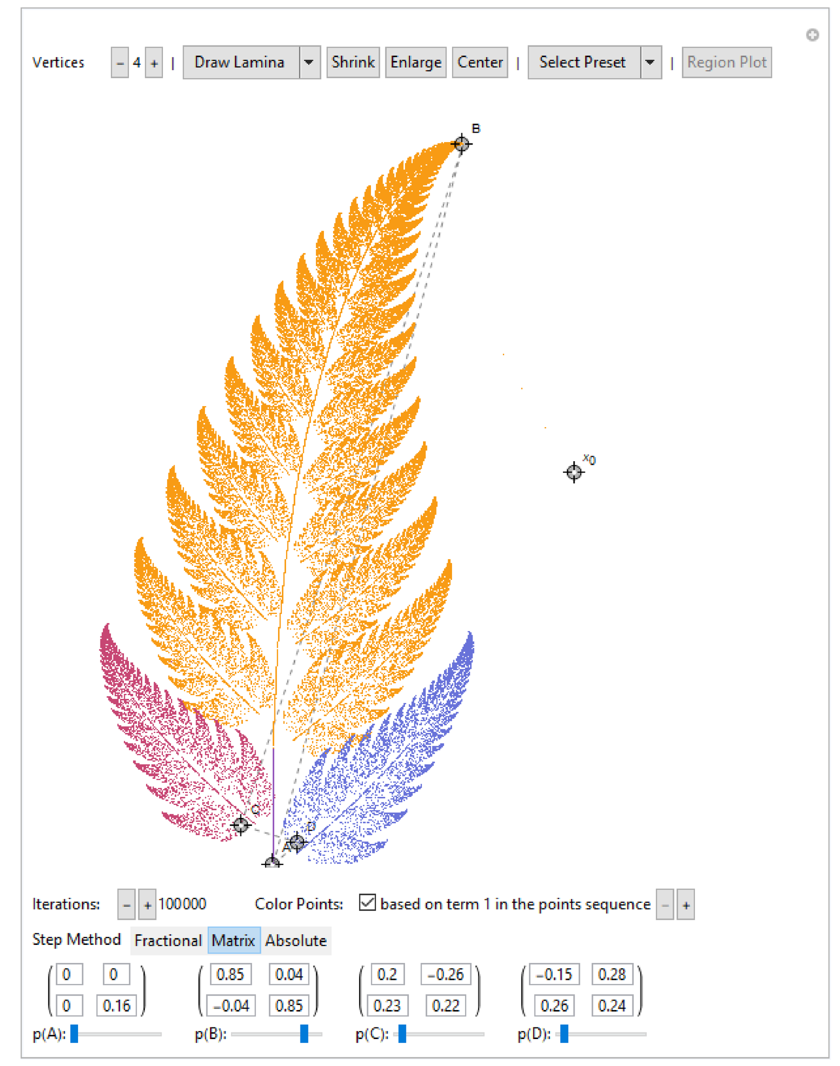
Where $\mathrm{H}_\alpha(A)$ denotes the **Rényi entropy** of order α.

Our software can approximate this dimension using regression techniques for arbitrary α. Below there is a series of Sierpiński-type fractals with the scale factor between −½ and ½. The plots illustrate the limit of $\dim_{Box} A$ as $\epsilon \to 0$. The orange line (where it's present) is the true Hausdorff dimension, the red line is the average value, and we have plotted our regression line in green. Note the difference near zero between 100 sparse points (top) and 10,000 points (bottom).



Features of Fractal Explorer GUI





- Select the input region from the Wolfram Knowledgebase Lamina Data.
- Describe geometric transformations as strings and view interpretation in homogeneous coordinates.
- Dynamically move points as locators or select from a preset convex n-gon.
- Different modes allow users to specify the IFS in terms of scale factor, transformation matrix or absolute step size.
- Translate IFS from the geometric region GUI to the chaos game GUI and vice versa.